ADDENDUM TO "ON SOME ALGEBRAIC PROPERTIES OF THE BESSEL POLYNOMIALS"(1)

BY

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In [2] it has been shown that the Galois groups G_n of the irreducible (over the rational field) Bessel polynomials $z_n(x)=\sum_{\nu=0}^{n}(n+\nu)!\{2^\nu(n-\nu)!\nu!\}^{-1}x^{n-\nu}$ are the symmetric groups on n symbols S_n, except, possibly, for $n=9$, 11 and 12. It also was shown that if G_9 contains an element of order 5 and if G_{11} and G_{12} each contain elements of order 7, then also G_9, G_{11} and G_{12} are the symmetric groups S_n with $n=9$, 11 and 12, respectively.

In order to prove the existence of elements of orders 5 and 7 respectively, one may use a classical theorem of Dedekind (see [3, p. 445]), the relevant part of which is the following:

If

$$(*)\qquad z_n(x)\equiv f_1(x)\cdot f_2(2)\cdot\cdots\cdot f_r(x)\pmod p\quad (p = \text{rational prime})$$

with $f_i(x)$ $(1\leqq i\leqq r)$ irreducible and incongruent mod p, then the Galois group of $z_n(x)$ contains at least one permutation of r cycles, each cycle corresponding to one of the factors in (*) and of the same order as the degree of the polynomial $f_i(x)$ to which it corresponds.

The following factorizations were obtained for $z_n(x)$ $(n=9, 11, 12)$:

$$z_9(x)\equiv(x+3)(x^3+12x^2-13x+3)(x^5+x^4-5x^3-x^2+7x+13)\pmod{29}$$

$$\begin{aligned} z_{11}(x)&\equiv(x^4+27x^3+14x^2+74x+23)\\ &\quad\cdot(x^7+39x^6+35x^5+121x^4+148x^3+60x^2+25x+53)\pmod{149}\end{aligned}$$

$$\begin{aligned} z_{12}(x)&\equiv(x^5+76x^4+65x^3+63x^2+29x+15)\\ &\quad\cdot(x^7+2x^6+27x^5+28x^4+30x^3+38x^2+15x+3)\pmod{89}.\end{aligned}$$

The first factorization was done by Dr. M. Newman and Mr. K. Kloss on the IBM 704 of the National Bureau of Standards in October 1961, the last two by Professor J. D. Brillhart and R. Stauduhar, using an algorithm of E. Berlekamp, on the CDC 6400 at the University of Arizona, in February 1969. (Required computer time: 4 and 6 seconds, resp.)

For $z_n(x)$, $n=9$, 11 or 12, denote by P_n the permutation whose existence is guaranteed by Dedekind's theorem. Then $P_9=(a, b, c)(d, e, f, g, h)$ (one symbol is left invariant) and $Q=P_9^3=(d, g, e, h, f)\in G_9$ and is of order 5; hence, $G_9\cong S_9$.

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Similarly, $P_{11}=(a, b, c, d)(e, f, g, h, i, j, k)$ and $Q'=P_{11}^4=(e, i, f, j, g, k, h) \in G_{11}$ and is of order 7, so that $G_{11} \cong S_{11}$. Finally, $P_{12}=(a, b, c, d, e)(f, g, h, i, j, k, l)$ and $Q''=P_{12}^5=(f, k, i, g, l, j, h) \in G_{12}$ and is of order 7, so that $G_{12} \cong S_{12}$. The details may be found in [1] and [2]. In conclusion and with previous notations one obtains, therefore the following

THEOREM. *The Galois groups G_n of the irreducible Bessel polynomials $z_n(x)$ satisfy $G_n \cong S_n$.*

The following conjecture is proven in [2] for special values of n (e.g., for $n \leqq 400$, $n=p^m$, $n=k \cdot p-1$ $(k<p)$, etc.):

CONJECTURE A. All Bessel polynomials are irreducible over the rational field.

For general n the problem is still open. By the present Theorem, the truth of Conjecture A immediately implies

CONJECTURE B. For all n, $G_n \cong S_n$.

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